Maths Challenge No. 4

31 October, 2022

Question 1. Let $\mathbf{e}_1, \mathbf{e}_2$ be any two linearly independent vectors in $\mathbb{R}^2$. This means that neither one of them can be written as a multiple of the other. Consider the set

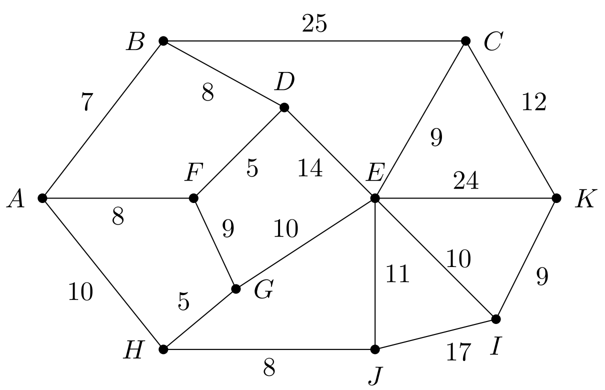
$$🎃 = \{n_1\mathbf{e}_1 + n_2\mathbf{e}_2 : n_1, n_2 \in \mathbb{Z}\}.$$

Such a set is called a 2D lattice, and (n_1, n_2) are called the coordinates of the point $n_1\mathbf{e}_1 + n_2\mathbf{e}_2 \in$ 🎃.

Pick a point $r = (x, y) \in \mathbb{R}^2$ uniformly at random. Find the coordinates of the point $p \in$ 🎃 that is closest to $r \in \mathbb{R}^2$. If there is more than one such point, then the algorithm should locate all of them.

Generalise this to three dimensions.

Question 2. Find the shortest path from A to K. The numbers on the edges denote the distance between the adjoining nodes.



[Note: There is no requirment to visit every vertex, or to traverse every edge.]

Question 3. Find a formula for the volume of a regular octahedron, given its side length. Find the ratio of the volume of a sphere to the volume of its inscribed regular octahedron.

[Note: For the normed space $(\mathbb{R}^3, \|\cdot\|)$, the closed ball generated by the ℓ_1 norm is a regular octahedron, and that generated by the ℓ_2 norm is a sphere.]

Question 4. Let $f(t)$ be a function defined over $\mathbb{R}$, and suppose its second derivative exists and is continuous. Show that, if $f(t)$ satisfies the functional equation

$$f(x+y)f(x-y) = f(x)f(x) + f(y)f(y) - 1, \forall x, y \in \mathbb{R},$$

then $f''(t) = \pm m^2 f(t)$, for some constant $m \geq 0$.

Deduce the explicit form of $f(t)$.

Visit `nustmaths.github.io` for solutions and previous question sets.